Particle Counting Noise

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1 The Particle Counting Noise

We define the Particle Counting noise as a random process $n_c(t)$, whose contribution corresponds to the difference between the measured particle concentration at the receiver location $\hat{c}_R(t)$ and the expected measured particle concentration $\langle \hat{c}_R(t) \rangle$, where $\langle . \rangle$ denotes the ensemble average operator:

$$n_c(t) = \hat{c}_R(t) - \langle \hat{c}_R(t) \rangle \quad (1)$$

The expected particle concentration rate $\langle \hat{c}_R(t) \rangle$ corresponds the true particle concentration $c_R(t)$ that we would measure at the receiver in the absence of noise:

$$\langle \hat{c}_R(t) \rangle = c_R(t) \epsilon \mathbb{R} \quad (2)$$

In other words, $n_c(t)$ is an unwanted perturbation on the particle concentration measured at the receiver location around its expected value $c_R(t)$ due to the particle counting noise.

In order to properly model the Particle Counting noise random process $n_c(t)$ we consider the following assumptions:

- the actual number of particles $\hat{N}_p(t)$ inside the receptor space increments/decrements its value when a particle enters/leaves the receptor space. Since particles are independent, these events can be supposed independent.
- The occurrence rate of particle entering/leaving the receptor space is proportional to the particle concentration at the receiver location $c(x_R, y_R, z_R, t)$, equal to the expected continuous particle concentration $c_R(t)$.

Under these assumptions, the resulting actual number of particles $\hat{N}_p(t)$ inside the receptor space is a volume Non-Homogeneous Poisson counting process, whose rate of occurrence corresponds to the expected particle concentration $c_R(t)$:

$$\hat{N}_p(t) \sim \text{Poiss}(c_R(t)) \tag{3}$$

According to Eq. (3) we can recover the PDF of the actual number of particles $\hat{N}_p(t)$ in the receptor space at time t, given the expected particle concentration $c_R(t)$:

$$Pr\left(\hat{N}_p(t) = m\right) = \frac{(c_R(t)(4/3)\pi\rho^3)^m}{m!} e^{-c_R(t)(4/3)\pi\rho^3} \tag{4}$$

According to the Poisson process in Eq. (3), the expected number of particles $\langle \hat{N}_p(t) \rangle$ contained in the receptor space can be computed by multiplying the volume Poisson process rate, which is the concentration $c_R(t)$, by the size of the receptor space $(4/3)\pi\rho^3$:

$$\langle \hat{N}_p(t) \rangle = c_R(t)\frac{4}{3}\pi\rho^3 \tag{5}$$

Its variance in the number of particles contained in the receptor space has the same value as $\langle \hat{N}_p(t) \rangle$:

$$\langle (\hat{N}_p(t) - \langle \hat{N}_p(t) \rangle)^2 \rangle = c_R(t)\frac{4}{3}\pi\rho^3 \tag{6}$$

The actual measured particle concentration $\hat{c}_R(t)$ corresponds to the actual number of particles $\hat{N}_p(t)$ divided by the size of the receptor space:

$$\hat{c}_R(t) = \frac{\hat{N}_p(t)}{(4/3)\pi\rho^3} \tag{7}$$

Therefore, the average $\langle \hat{c}_R(t) \rangle$ of the actual measured particle concentration is equal to the expected particle concentration $c_R(t)$:

$$\langle \hat{c}_R(t) \rangle = c_R(t) \tag{8}$$

The variance of the actual measured particle concentration is equal to the expected particle concentration $c_R(t)$ divided by the size of the receptor space:

$$\langle (\hat{c}_R(t) - \langle \hat{c}_R(t) \rangle)^2 \rangle = \frac{c_R(t)}{(4/3)\pi\rho^3} \tag{9}$$

Given Eq. (1) and Eq. (2), the random process $n_c(t)$ has zero average value and the RMS of the perturbation $n_c(t)$ on the actual measured particle concentration $\hat{c}_R(t)$ is:

$$\mathrm{RMS}(n_c(t)) = \sqrt{\langle(\hat{c}_R(t) - \langle\hat{c}_R(t)\rangle)^2\rangle} = \sqrt{\frac{c_R(t)}{(4/3)\pi\rho^3}} \tag{10}$$

It is possible to reduce the value of $\mathrm{RMS}(n_c(t))$ through averaging in time a number M of measures of the particle concentration $\hat{c}_R(t)$:

$$\hat{c}_R(t) = \frac{1}{M}\sum_{m=1}^{M}\hat{c}_R(t - t_m) \tag{11}$$

The best results in terms of noise are obtained when the M measures are statistically independent. For this, we assume independent measures when they are taken at time instants spaced by an interval τ_p, as defined in [1]. Then, if we also assume to have a quasi-constant expected concentration in a time interval τ (which means that the bandwidth of the signal $c_R(t)$ is less than $1/\tau$), then the maximum value of M is equal to the time interval τ divided by τ_p:

$$M = \frac{\tau}{\tau_p} \tag{12}$$

thus, reducing the RMS of the perturbation $\mathrm{RMS}(n_c(t))$ by a factor $\sqrt{M}$:

$$\mathrm{RMS}(n_c(t)) = \sqrt{\frac{c_R(t)}{(4/3)\pi\rho^3 M}} \tag{13}$$

The waiting time τ_p corresponds to the average time required for a particle to leave the reception space. τ_p is equal to the average distance to the spherical boundary, divided by the velocity of a particle v_p. The average distance corresponds to the receptor space radius ρ:

$$\tau_p = \frac{\rho}{v_p} \tag{14}$$

The velocity v_p of a particle comes from the first Fick's law of diffusion [3,5]. For this, the particle concentration flux $\bar{J}(\bar{x}, t)$ at time instant t and location

$\bar{x}$, is equal to the spatial gradient (operator ∇) of the particle concentration $c(\bar{x},t)$ multiplied by the diffusion coefficient D:

$$\bar{J}(\bar{x},t) = -D\nabla c(\bar{x},t) \tag{15}$$

When we have homogeneous concentration $\bar{c}$ inside the receptor space and zero concentration outside the receptor space, $\nabla c(\bar{x},t)$ is equal to the opposite $-\bar{c}$ of the concentration divided by the radius ρ of the receptor space. Further, the particle concentration flux $\bar{J}(\bar{x},t)$ is equal, by definition, to the particle concentration $\bar{c}$ multiplied by the particle velocity v_p. If we solve Eq. (15) for the particle velocity, we obtain:

$$v_p = \frac{D}{\rho} \tag{16}$$

The average time τ_p is therefore equal to the radius ρ squared and divided by the diffusion coefficient D:

$$\tau_p = \frac{\rho^2}{D} \tag{17}$$

which is in agreement with the results from [1,2]. The final expression for the RMS of the perturbation $\mathrm{RMS}(n_c(t))$ becomes:

$$\mathrm{RMS}(n_c(t)) = \sqrt{\frac{c_R(t)}{(4/3)\pi D\rho\tau}} \tag{18}$$

where $c_R(t)$ is the expected measured particle concentration, D is the diffusion coefficient, ρ is the radius of the receptor space and τ is the time interval in which we expect a quasi-constant particle concentration.

According to [6], the relation between the input particle concentration rate $\hat{r}_T(t)$ and the measured particle concentration $c_R(t)$ at the receiver location is expressed in the frequency (f) domain as:

$$\tilde{\mathbf{c}}_R(f) = \tilde{\mathbf{B}}(f)\tilde{\hat{r}}_T(f) \tag{19}$$

where $\tilde{\hat{r}}_T(f)$ and $\tilde{\mathbf{c}}_R(f)$ are the Fourier transforms [4] of the particle concentration rate $\hat{r}_T(t)$ and the particle concentration $c_R(t)$, respectively. $\tilde{\mathbf{B}}(f)$ is the Transfer Function Fourier Transform [4] (TFFT) of the propagation module. The same relation in the time (t) domain becomes:

$$c_R(t) = b(t) * \hat{r}_T(t) \tag{20}$$

where $*$ denotes the convolution operator [4], $b(t)$ is the impulse response of the propagation module and $\hat{r}_T(t)$ is the input particle concentration rate. The formula for the RMS of the perturbation $\mathrm{RMS}(n_c(t))$ on the signal $\hat{c}_R(t)$ becomes:

$$\mathrm{RMS}(n_c(t)) = \sqrt{\frac{b(t) * \hat{r}_T(t)}{(4/3)\pi D\rho\tau}} \tag{21}$$

where D is the diffusion coefficient, ρ is the radius of the spherical receptor space, and τ is the time in which we expected a quasi-constant particle concentration.

References

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